A note on the transformation coefficients between the standard and non-standard representations

1982 J. Phys. A: Math. Gen. 15 2013

(http://iopscience.iop.org/0305-4470/15/7/013)

J. Phys. A: Math. Gen. 15 (1982) 2013–2016. Printed in Great Britain

A note on the transformation coefficients between the standard and non-standard representations

Ch Suryanarayana† and M Kondala Rao
Department of Applied Mathematics, Andhra University, Waltair 530 003, India

Received 9 December 1981

Abstract. In a physically significant case, closed formulae for the transformation coefficients between the standard and non-standard representations of the symmetric group S_N are obtained and the results are explicitly exhibited.

1. Introduction

The symmetric group S_N finds a place in the study of systems consisting of N identical particles as a part of the symmetry group of the Hamiltonian. Hence, acknowledgement of the irreducible representations (IR) of S_N is of great help in the investigation of such systems. There may, in general, be systems consisting of two sets of N_1 and N_2 particles, each in a state which transforms according to an IR Γ_1 of S_{N_1} and IR Γ_2 of S_{N_2}. These states may be two different shells in an atomic or nuclear shell model or two different clusters in a nuclear cluster model (Kramer 1968). In such cases one has to adapt the total system of N ($N = N_1 + N_2$) particles to an overall symmetry. This requires the study of the representations of the symmetric group, reduced with respect to its subgroups arbitrarily. Such representations are known as non-standard representations (Kaplan 1962). Horie (1964) studied extensively this problem as part of the calculation of fractional parentage coefficients. Kaplan (1962) introduced the transformation matrix which performs the transition between the representations with different types of reduction and obtained the matrix for the transition from the standard Young–Yamanouchi basis to a basis corresponding to the representation reduced with respect to a subgroup $S_{N_1} \times S_{N_2}$. Derivation of the matrix in the case of more complicated types of reduction is also given by him.

Horie (1964) described a different recursion method to obtain the transformation matrix. He derived compact formulae, in terms of axial distances τ_{ij}, for the matrices which perform the transition from the canonical chain to the chain $S_N \supset S_{N-m} \times S_m$ in the two particular cases namely $[m]$, the total symmetric representation and $[1^m]$, the antisymmetric representation of S_m. To get a closed formula for the transformation matrix in the case of general representation is complicated because of the multidimensionality of the representation and the multiplicity of the reduction $S_N \supset S_{N-m} \times S_m$. The above problem is considered in the present paper, in the case of multidimensional representations of the type $[2^a 1^{n-2a}]$ which are physically significant.

† Present address: Department of Applied Mathematics, Andhra University P G Extension Centre, NUZVID 521201, AP, India.

0305-4470/82/072013+04\$02.00

2. Transformation coefficients between the standard and non-standard representations

The representation $\Gamma^{[\lambda]}$ of the symmetric group S_N reduced with respect to the chain of subgroups $S_N \supset S_{N-1} \supset \ldots \supset S_2$ is known as the standard representation of S_N. In other words the representation provided by the standard Young tableaux (or the Yamanouchi symbols) is the standard representation of S_N. In a number of problems in quantum mechanics it is convenient to work with basis functions for a representation $\Gamma^{[\lambda]}$ which are characterised by a definite permutation symmetry with respect to the permutations of both the first N_1 and the last N_2 ($N = N_1 + N_2$) numbers. Functions such as these form a basis for a representation that splits into irreducible components upon passing from the group S_N to its subgroup $S_{N_1} \times S_{N_2}$. In contrast to the standard representation, which is reduced with respect to the subgroups $S_{N-1} \supset S_{N-2} \supset \ldots \supset S_2$, a representation which is reduced with respect to a different set of subgroups is said to be non-standard (Elliot *et al* 1953).

In studying the clustering properties of many-particle systems and in the calculations of the reduced widths or the spectroscopic factors for nucleon clusters, by shell model, the fractional parentage coefficients (FPC) which separate more than one particle are quite important. To obtain these FPC, it is necessary to find the transformation coefficients between the standard and non-standard representations of the symmetric group. Kaplan (1962) and Horie (1964) derived formulae to obtain these coefficients. Horie (1964) presented closed formulae for these transformation coefficients in the two special cases—$[m]$ and $[1^m]$ of S_m, in the chain $S_n \supset S_{n-m} \times S_m$. These two representations considered by Horie are one dimensional and therefore the reduction $[\lambda] \to [\lambda'] \times [\lambda'']$ trivially becomes multiplicity free. The multi-dimensional representations $[\lambda'']$ are complicated, but nevertheless, are worth considering.

Table 1. Transformation coefficients in the chain $S_7 \supset S_4 \times S_3$.

$[\lambda]$	$[21^5]$		$[2^2 1^3]$						$[2^3 1]$			
r'	$[1^4]$		$[21^2]_1$		$[21^2]_2$		$[21^2]_3$		$[2^2]_1$		$[2^2]_2$	
$r^{(i)}$ \ r''	$[21]_1$	$[21]_2$	$[21]_1$	$[21]_2$	$[21]_1$	$[21]_2$	$[21]_1$	$[21]_2$	$[21]_1$	$[21]_2$	$[21]_1$	$[21]_2$
$r^{(1)}$	0	0	0	0	0	0	0	0	1	0	0	0
$r^{(2)}$	0	0	$\sqrt{3}/3$	$-1/3$	0	0	0	0	0	1	0	0
$r^{(3)}$	0	0	$\sqrt{6}/3$	$\sqrt{2}/6$	0	0	0	0	0	0	0	0
$r^{(4)}$	$\sqrt{10}/5$	$-\sqrt{30}/15$	0	$\sqrt{30}/6$	0	0	0	0	0	0	0	0
$r^{(5)}$	$\sqrt{15}/15$	$2\sqrt{5}/15$	0	0	0	0	0	0	0	0	0	0
$r^{(6)}$	0	$\sqrt{7}/3$	0	0	$\sqrt{3}/3$	$-1/3$	0	0	0	0	1	0
$r^{(7)}$	—	—	0	0	$\sqrt{6}/3$	$\sqrt{2}/6$	0	0	0	0	0	1
$r^{(8)}$	—	—	0	0	0	$\sqrt{30}/6$	0	0	0	0	0	0
$r^{(9)}$	—	—	0	0	0	0	$\sqrt{3}/3$	$-1/3$	0	0	0	0
$r^{(10)}$	—	—	0	0	0	0	$\sqrt{6}/3$	$\sqrt{2}/6$	0	0	0	0
$r^{(11)}$	—	—	0	0	0	0	0	$\sqrt{30}/6$	0	0	0	0
$r^{(12)}$	—	—	0	0	0	0	0	0	0	0	0	0
$r^{(13)}$	—	—	0	0	0	0	0	0	0	0	0	0
$r^{(14)}$	—	—	0	0	0	0	0	0	0	0	0	0

In the LS coupling scheme, the orbital and spin wavefunctions of a system of n particles are treated separately. The total wavefunction will be a product of the space and spin wavefunctions. As a consequence of Pauli's exclusion principle, the Young diagram for the orbital wavefunction should not have more than two cells in any row. Hence, such physically important representations of S_N only are considered here. Let the representation $[\lambda]$ of S_n be of the form $[2^a 1^{n-2a}]$. Then, in general, we may choose $[\lambda'] = [2^b 1^{n-m-2b}]$ and $[\lambda''] = [2^c 1^{m-2c}]$. One may note that the decomposition $[\lambda] \to [\lambda'] \times [\lambda'']$ is not always multiplicity free. But, it may be observed that in the present case it is necessary that $a = b$ or $a = b+1$ for the decomposition to be multiplicity free. It is interesting to note that when $a = b$ the problem reduces to the one considered by Horie. Having noticed this, the case $a = b+1$, which implies $c = 1$, is studied and closed formulae are obtained following Horie. As the derivation of these formulae is essentially similar to that of Horie (1964), the final formulae alone have been presented. Since the representations considered by Horie are one dimensional, the same symbols $[m]$ and $[1^m]$ have been used both for the representations and their Yamanouchi symbols in writing the transformation coefficients. But, as the multi-dimensional representations are considered here, a suffix i is attached to the representation symbol denoting the ith Yamanouchi symbol in that representation. The formulae for transformation coefficients (the notation adopted is the same as that of Horie) are:

for $1 < q < m-1$

$$\langle r'\rho | r', [2\,1^{m-2}]_q \rangle = \left[\frac{1}{q!} \prod_{i>j} \left(1 - \frac{1}{\tau_{i-m+qj-m+q}}\right) - \sum_{k=q+1}^{m-1} \frac{1}{k!} \prod_{i>j} \left(1 - \frac{1}{\tau_{i-m+kj-m+k}}\right) - \frac{1}{m!} \prod_{i>j} \left(1 + \frac{1}{\tau_{ij}}\right) \right]^{1/2} \varepsilon$$

where

$$\varepsilon = \begin{cases} -1 & \text{if } r_{n-m+q} \leq r_{n-m+q+1} \leq \ldots \leq r_n, \\ 1 & \text{otherwise;} \end{cases}$$

for $q = 1$

$$\langle r'\rho | r', [2\,1^{m-2}]_1 \rangle = \left[\frac{1}{2}\left(1 + \frac{1}{\tau_{n-m+2n-m+1}}\right) - \sum_{k=3}^{m} \frac{1}{k!} \prod_{i>j} \left(1 + \frac{1}{\tau_{i-m+kj-m+k}}\right) \right]^{1/2};$$

for $q = m-1$

$$\langle r'\rho | r', [2\,1^{m-2}]_{m-1} \rangle = \left[\frac{1}{(m-1)!} \prod_{i>j} \left(1 - \frac{1}{\tau_{i-1j-1}}\right) - \frac{1}{m!} \prod_{i>j} \left(1 - \frac{1}{\tau_{ij}}\right) \right]^{1/2} \varepsilon$$

where

$$\varepsilon = \begin{cases} (-1)^m & \text{if } r_{n-m+1} \leq r_{n-m+2}, \\ (-1)^{m+1} & \text{otherwise.} \end{cases}$$

In all the above formulae i and j take values from $n-m+1$ to n and τ_{ij} is the axial distance between the numbers i and j in the Young tableau i.e. $\tau_{ij} = \lambda(r_i) - \lambda(r_j) - (r_i - r_j)$.

These formulae are employed to obtain the transformation coefficients in the chains $S_7 \supset S_4 \times S_3$ and $S_7 \supset S_3 \times S_4$ and the results are presented in tables 1 and 2 respectively. These results are found to agree very well with those worked out by Innaiah (1980) using Kaplan's (1962) method.

Table 2. Transformation coefficients in the chain $S_7 \supset S_3 \times S_4$.

$[\lambda]$	$[21^5]$			$[2^2 1^3]$					
r'	$[1^3]$			$[21]_1$			$[21]_2$		
$r^{(i)}$ \ r''	$[21^2]_1$	$[21^2]_2$	$[21^2]_3$	$[21^2]_1$	$[21^2]_2$	$[21^2]_3$	$[21^2]_1$	$[21^2]_2$	$[21^2]_3$
$r^{(1)}$	0	0	0	1/2	$-\sqrt{3}/6$	$\sqrt{6}/12$	0	0	0
$r^{(2)}$	0	0	0	$\sqrt{3}/2$	1/6	$-\sqrt{2}/12$	0	0	0
$r^{(3)}$	$\sqrt{6}/4$	$-\sqrt{2}/4$	1/4	0	$2\sqrt{2}/3$	1/12	0	0	0
$r^{(4)}$	$\sqrt{10}/4$	$\sqrt{30}/20$	$-\sqrt{15}/20$	0	0	$\sqrt{15}/4$	0	0	0
$r^{(5)}$	0	$2\sqrt{5}/5$	$\sqrt{10}/20$	0	0	0	1/2	$-\sqrt{3}/6$	$\sqrt{6}/12$
$r^{(6)}$	0	0	$\sqrt{350}/20$	0	0	0	$\sqrt{3}/2$	1/6	$-\sqrt{2}/12$
$r^{(7)}$	—	—	—	0	0	0	0	$2\sqrt{2}/3$	1/12
$r^{(8)}$	—	—	—	0	0	0	0	0	$\sqrt{15}/4$
$r^{(9)}$	—	—	—	0	0	0	0	0	0
$r^{(10)}$	—	—	—	0	0	0	0	0	0
$r^{(11)}$	—	—	—	0	0	0	0	0	0
$r^{(12)}$	—	—	—	0	0	0	0	0	0
$r^{(13)}$	—	—	—	0	0	0	0	0	0
$r^{(14)}$	—	—	—	0	0	0	0	0	0

Acknowledgments

The authors wish to thank Dr T S G Krishna Murty and Dr L S R K Prasad for many useful discussions during the preparation of the paper.

References

Elliot J P, Hope J and Jahn H A 1953 *Phil. Trans. R. Soc.* A **246** 241–69
Horie H 1964 *J. Phys. Soc. Japan* **19** 1783–99
Innaiah P 1980 *MPhil thesis*, Andhra University, Waltair, India
Kaplan I G 1962 *Sov. Phys.-JETP* **14** 401
—— 1975 *Symmetry of Many Electron Systems* (New York: Academic)
Kramer P 1968 *Z. Phys.* **216** 68